**3B-2** Find a $\Sigma$ notation expression for

~~a)~~ $3 - 5 + 7 - 9 + 11 - 13$

~~b)~~ $1 + 1/4 + 1/9 + \cdots + 1/n^2$

c) $\sin x/n + \sin(2x/n) + \cdots + \sin((n-1)x/n) + \sin x$

14/8/25

a) $\displaystyle\sum_{i=1}^{6} (-1)^{i+1} (1+2i)$

$\displaystyle= \sum_{i=1}^{6} (-1)^{i} (-1-2i)$

b) $\displaystyle\sum_{i=1}^{n} \frac{1}{i^2}$

**3B-3** Write the upper, lower, left and right Riemann sums for the following integrals, using 4 equal subintervals:

a) $\int_0^1 x^3 dx$ ~~b)~~ $\int_{-1}^3 x^2 dx$ c) $\int_0^{2\pi} \sin x dx$

b) $\Delta x = \frac{b-a}{n}$

$= \frac{3-(-1)}{4}$

$= 1$

right

$c_i = -1 + i\Delta x$

$f(c_i) = (i-1)^2$

$\int_{-1}^{3} x^2 dx \approx \sum_{i=1}^{4} f(c_i)\Delta x$

$= 0 + 1 + 4 + 9$

$= 14$

left

$c_i = -1 + (i-1)\Delta x$

$= i - 2$

$f(c_i) = (i-2)^2$

$\int_{-1}^{3} x^2 dx \approx \sum_{i=1}^{4} f(c_i)\Delta x$

$= 1 + 0 + 1 + 4$

$= 6$

upper

$c_1 = -1,\ c_2 = 1,\ c_3 = 2,\ c_4 = 3$

$\int_{-1}^{3} x^2 dx \approx \sum_{i=1}^{4} f(c_i)\Delta x$

$= (-1)^2 + 1^2 + 2^2 + 3^2$

$= 15$

lower

$c_1 = 0,\ c_2 = 0,\ c_3 = 1,\ c_4 = 2$

$\int_{1}^{3} x^2 dx \approx \sum_{i=1}^{4} f(c_i)\Delta x$

$= 0^2 + 0^2 + 1^2 + 2^2$

$= 5$

**3B-4** Calculate the difference between the upper and lower Riemann sums for the following integrals with $n$ intervals

~~a)~~ $\int_0^b x^2 dx$ b) $\int_0^b x^3 dx$

a) $\Delta x = \frac{b-0}{n} = \frac{b}{n}$

lower

$c_i = (i-1)\Delta x = \frac{b}{n}i - \frac{b}{n}$

$f(c_i) = \frac{b^2}{n^2}(i-1)^2$

$\sum_{i=1}^{n} f(c_i)\Delta x$

$= \sum_{i=1}^{n} \left(\frac{b}{n}\right)^2 (i-1)^2 \left(\frac{b}{n}\right)$

$= \left(\frac{b}{n}\right)^3 \sum_{i=1}^{n} (i-1)^2$

upper

$c_i = i\Delta x = \frac{b}{n}i$

$f(c_i) = \left(\frac{b}{n}\right)^2 i^2$

$\sum_{i=1}^{n} f(c_i)\Delta x$

$= \sum_{i=1}^{n} \left(\frac{b}{n}\right)^2 i^2 \left(\frac{b}{n}\right)$

$= \left(\frac{b}{n}\right)^3 \sum_{i=1}^{n} i^2$

Difference

$= \left(\frac{b}{n}\right)^3 \sum_{i=1}^{n} i^2 - \left(\frac{b}{n}\right)^3 \sum_{i=1}^{n} (i-1)^2$

$= \left(\frac{b}{n}\right)^3 \sum_{i=1}^{n} \left(i^2 - (i-1)^2\right)$

$= \left(\frac{b}{n}\right)^3 \sum_{i=1}^{n} (2i-1)$

$= \left(\frac{b}{n}\right)^3 \left(2\sum_{i=1}^{n} i - n\right)$

$= \left(\frac{b}{n}\right)^3 \left(2\left(\frac{n(n+1)}{2}\right) - n\right)$

$= \left(\frac{b}{n}\right)^3 (n^2 + n - n)$

$= \frac{b^3}{n}$

**3B-5** Evaluate the limit, by relating it to a Riemann sum.

$$\lim_{n\to\infty} \frac{\sin(b/n) + \sin(2b/n) + \cdots + \sin((n-1)b/n) + \sin(nb/n)}{n}$$

$$\Delta x = \frac{1}{n}$$

$$f(c_i) = \sin\left(i\frac{b}{n}\right)$$

$$\sin\left(\frac{b}{n}\right) + \sin\left(2\frac{b}{n}\right) + \cdots + \sin\left((n-1)\frac{b}{n}\right) + \sin\left(n\frac{b}{n}\right)$$

$$= \sum_{i=1}^{n} \sin\left(i\frac{b}{n}\right)$$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(i\frac{b}{n}\right)\Delta x$$

$$= \int_0^1 \sin bx \, dx$$

$$= -\frac{\cos bx}{b}\Big|_0^1$$

$$= -\frac{1}{b}(\cos b - \cos 0)$$

$$= \frac{1 - \cos b}{b}$$

**4J-1** Suppose it takes $k$ units of energy to lift a cubic meter of water one meter. About how much energy $E$ will it take to pump dry a circular hole one meter in diameter and 100 meters deep that is filled with water? (Give reasoning.)

$$\Delta V = A \Delta x$$

$$\Delta E = k x_i A \Delta x$$

$$\Rightarrow dE = k x A dx$$

$x$, height in metres

$f(x)$, volume in $m^3$ per metre

$$E = \int_0^1 k x (1 \times 1)\, dx$$

$$\int_0^1 f(x)\, dx = k$$

$$= k \frac{x^2}{2} \Big|_0^1$$

$$= \frac{k}{2}$$

$\therefore$ $\frac{k}{2}$ units of energy required to lift a cubic meter of water out of a circular hole 1 meter deep.

$$E = \int_0^{100} k x \pi r^2\, dx$$

$$= \int_0^{100} k \pi x (0.5)^2\, dx$$

$$= k\pi \int_0^{100} \frac{x}{4}\, dx$$

$$= k\pi \left( \frac{x^2}{8} \right) \Big|_0^{100}$$

$$= k\pi \left( \frac{10000}{8} - 0 \right)$$

$$= \frac{10000}{8} k\pi$$

**3C-1** Find the area under the graph of $y = 1/\sqrt{x-2}$ for $3 \le x \le 6$

**3C-2** Calculate

$\cancel{a})\ \int_0^2 \sqrt{3x+5}\,dx$ b) $\int_0^2 (3x+5)^n dx$ c) $\int_{3\pi/4}^{\pi} \frac{\sin x dx}{\cos^3 x}$

1. $y = \frac{1}{\sqrt{x-2}}$, $3 \le x \le 6$

$$\text{Area} = \int_3^6 y\,dx$$

$$= \int_3^6 \frac{1}{\sqrt{x-2}}\,dx$$

$$= \left.\frac{\sqrt{x-2}}{1/2\,(1)}\right|_3^6$$

$$= 2\sqrt{x-2}\,\Big|_3^6$$

$$= 2\sqrt{4} - 2\sqrt{1}$$

$$= 2$$

2. a) $\int_0^2 \sqrt{3x+5}\,dx$

$$= \left.\frac{(3x+5)^{3/2}}{3/2 \cdot (3)}\right|_0^2$$

$$= \frac{2}{9}(3x+5)^{3/2}\Big|_0^2$$

$$= \frac{2}{9}\left(11^{3/2} - 5^{3/2}\right)$$

**3C-3** Calculate

$\cancel{a})\ \displaystyle\int_1^2 \frac{xdx}{x^2+1}$ b) $\displaystyle\int_b^{2b} \frac{xdx}{x^2+b^2}$

a) $$\int_1^2 \frac{x}{x^2+1}\,dx$$

$$= \frac{1}{2}\int_1^2 \frac{2x}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln(x^2+1)\Big|_1^2$$

$$= \frac{1}{2}(\ln 5 - \ln 2)$$

$$= \frac{1}{2}\ln\frac{5}{2}$$

does this integral describe? **3C-5** Find the area

~~a)~~ under one arch of $\sin x$.

b) under one arch of $\sin ax$ for a positive constant $a$.

a) $y = \sin x$

one arch $\Rightarrow$ 0 to $\pi$

$$\Rightarrow \text{Area} = \int_0^{\pi} \sin x \, dx$$

$$= -\cos x \Big|_0^{\pi}$$

$$= -(\cos\pi - \cos 0)$$

$$= 2$$

**3E-6** By comparing the given integral with an integral that is easier to evaluate. establish each of the following estimations:

a) $\int_0^1 \frac{dx}{1+x^3} > 0.65$ $\quad$ ~~b)~~ $\int_0^{\pi} \sin^2 x\, dx < 2$ $\quad$ ~~c)~~ $\int_{10}^{20} \sqrt{x^2+1}\, dx > 150$

b) $\int_0^{\pi} \sin^2 x\, dx$

$f(x) = \sin^2 x$

$g(x) = \sin x$

$f(x) - g(x)$

$= \sin^2 x - \sin x$

$= \sin x (\sin x - 1)$

$\because$ At $0 < x < \pi$, $0 < \sin x < 1$

$\Rightarrow -1 < \sin x - 1 < 0$

$\Rightarrow -1 < \sin x (\sin x - 1) < 0$

$\Rightarrow f(x) - g(x) < 0$

$\therefore f(x) < g(x)$ at $0 < x < \pi$

$\because f(x) = \sin^2 x < g(x) = \sin x$

and $0 < \pi$.

$\Rightarrow \int_0^{\pi} \sin^2 x\, dx < \int_0^{\pi} \sin x\, dx$

$\therefore \int_0^{\pi} \sin^2 x\, dx < 2$

c) Let $f(x) = \sqrt{x^2+1}$ and $g(x) = x$.

$(\sqrt{x^2+1})^2 = x^2 + 1 > x^2$

$\Rightarrow \sqrt{x^2+1} > x$

$\because 20 > 10$

$\Rightarrow \int_{10}^{20} \sqrt{x^2+1}\, dx > \int_{10}^{20} x\, dx$

$\therefore \int_{10}^{20} \sqrt{x^2+1}\, dx > 150$

$\int_{10}^{20} x\, dx$

$= \frac{x^2}{2} \Big|_{10}^{20}$

$= \frac{1}{2}(400 - 100)$

$= 150$

**4J-2** The amount $x$ (in grams) of a radioactive material declines exponentially over time (in minutes), according to the law $x = x_0e^{-kt}$, where $x_0$ is the amount initially present at time $t = 0$. If one gram of the material produces $r$ units of radiation/minute, about how much radiation $R$ is produced over one hour by $x_0$ grams of the material? (Give reasoning.)

$$x(t) = x_0 e^{-kt}$$

When $x_0 = 1$, $r \cdot e^{-kt_i} \Delta t$ of radiation is produced at interval $[t_i, t_i + \Delta t]$.

When $x_0 = x_0$,

$$R = \lim_{n\to\infty} \sum_{i=1}^{n} r \cdot x_0 e^{-kt_i} \Delta t$$

$$= \int_0^{60} r \cdot x_0 e^{-kt}\, dt$$

$$= -\frac{r x_0}{k} \left(e^{-kt}\right)\Big|_0^{60}$$

$$= -\frac{r x_0}{k} \left(e^{-60k} - 1\right)$$

$$= \frac{r x_0}{k} \left(1 - e^{-60k}\right)$$